Analytic properties of finite-temperature self-energies

H. Arthur Weldon
Department of Physics, West Virginia University, Morgantown, West Virginia 26506-6315
(Received 24 December 2001; published 5 April 2002)

The analytic properties in the energy variable k_0 of finite-temperature self-energies are investigated. A typical branch cut results from n particles being emitted into the heat bath and n' being absorbed from the heat bath. There are three main results: First, in addition to the branch points at which the cuts terminate, there are also branch points attached to the cuts along their length. Second, branch points at $k_0=\pm k$ are ubiquitous and for massive particles they are essential singularities. Third, in a perturbative expansion using free particle propagators or in a resummed expansion in which the propagator pole occurs at a real energy, the self-energy will have a branch point at the pole location.

DOI: 10.1103/PhysRevD.65.076010 PACS number(s): 11.10.Wx, 11.55.Bq, 12.38.Mh

I. INTRODUCTION

At non-zero temperatures most examinations of the self-energy have emphasized one-loop results. For massless gauge theories Braaten and Pisarski showed that all diagrams with one loop (but any number of external lines) will produce effects as large as the tree diagrams and must therefore be resummed [1,2]. In a resummed expansion the discontinuities of various one-loop and two-loop diagrams have been computed in QCD in order to predict processes relevant for quark gluon plasmas such as dilepton production, real photon production, and vector meson production.

There have been few investigations of the analytic properties of finite-temperature self-energies [3,4] and much of the emphasis has been on the behavior at zero four-momentum [5].

The development of the questions investigated in this paper and of the approach is best illustrated by considering how a familiar zero-temperature calculation changes at non-zero temperature.

$T=0$ example

A typical example arises for a massive, self-interacting scalar field with $\mathcal{H}_I=g^2\phi^4/4!$. One of the two-loop contributions to the zero-temperature self-energy is

$$\Pi_F(K)=-ig^4\int\frac{d^4P_1}{(2\pi)^4}\frac{d^4P_2}{(2\pi)^4}\frac{1}{(P_1^2-m^2+i\epsilon)}\times\frac{1}{(P_2^2-m^2+i\epsilon)}\frac{1}{(P_3^2-m^2+i\epsilon)},$$

where $P_3=K-P_1-P_2$. Direct integration of the energy variables P_{01} and P_{02} gives

$$\Pi_F(K)=-g^2\int\frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\frac{1}{2E_1 2E_2 2E_3}\times\left[\frac{1}{k_0-E_1-E_2-E_3+i\eta}-\frac{1}{k_0+E_1+E_2+E_3-i\eta}\right],\qquad(1.1)$$

with $E_j=(p_j^2+m^2)^{1/2}$. The first denominator produces a branch point in the self-energy at $k_0=3E(k/3)=(k^2+9m^2)^{1/2}$ and a branch cut along the positive k_0 axis for $3E(k/3)\leqslant k_0\leqslant\infty$. The second denominator produces a branch cut for $-\infty\leqslant k_0\leqslant -3E(k/3)$. From the operator point of view these two contributions arise from inserting a three-particle intermediate state in the two time-orderings contained in $\langle 0|T(\phi^3(x)\phi^3(y))|0\rangle$. At higher orders in perturbation theory the self-energy continues to be real and analytic in the open interval $-3E(k/3)<k_0<3E(k/3)$.

The same example at $T\neq 0$

At non-zero temperatures there are four real-time propagators organized into a 2×2 matrix D_{ij} with $i,j=1,2$ [6,7]. The proper self-energy becomes a matrix Π_{ij}. In the same $g^2\phi^4/4!$ theory the two-loop contribution to the time-ordered self-energy Π_{11} is

$$\Pi_{11}(K)=-ig^4\int\frac{d^4P_1}{(2\pi)^4}\frac{d^4P_2}{(2\pi)^4}\times D_{11}(P_1)D_{11}(P_2)D_{11}(P_3).$$

The finite-temperature propagators are

$$D_{11}(P)=\frac{1+f_{BE}}{P^2-m^2+i\epsilon}-\frac{f_{BE}}{P^2-m^2-i\epsilon},$$

where $f_{BE}=[\exp(\beta|p_0|)-1]^{-1}$ is the Bose-Einstein function. Performing the integrations over P_{01} and P_{02} leads to an integrand much more complicated than in Eq. (1.1). The new integrand can be expressed as a linear combination of eight terms each of which has a different k_0 dependence in the denominator. The denominators are of the form $k_0\pm E_1\pm E_2\pm E_3+i\eta(\vec{p}_j)$, in which all possible sign combinations occur. In the various denominators the sign of the infinitesimal imaginary parts are momentum dependent, which makes the calculation tedious and the analytic properties obscure. The complications of the $i\eta$'s is a result of the absolute value bars in f_{BE}.

A much simpler analytic structure is enjoyed by the retarded self-energy Π_R. All four Π_{ij} can be expressed in

terms of Π_R and Π_A, where $\Pi_A(K)=\Pi_R(-K)$. All four propagators D_{ij} can be expressed in terms of the retarded and advanced propagators D_R and D_A [8]. For example, $\Pi_{11}(K)$ is can be expressed as

$$\Pi_{11}(K)=[1+f]\Pi_R(K)-f\Pi_A(K),$$

where $f=[\exp(\beta k_0)-1]^{-1}$ has no absolute value bars and is an analytic function of k_0. The retarded and advanced self-energies have simple analytic structure: $\Pi_R(K)$ is holomorphic for $\mathrm{Im}(k_0)>0$ and $\Pi_A(K)$ is holomorphic for $\mathrm{Im}(k_0)<0$.

To compute the retarded self-energy directly without using the Π_{ij}, one can either use the real-time Feynman rules expressed in terms of D_R and D_A [8,9] or use the imaginary-time Feynman rules [10] and then analytically continue in energy. For the above two-loop example in $g^2\phi^4/4!$ theory the result for the retarded self-energy can be expressed as a sum of eight contributions:

$$\Pi_R(K)=\int\frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\sum_{A=1}^{8}\frac{f_A(\vec{p}_1,\vec{p}_2,\vec{p}_3)}{k_0+i\eta-\psi_A},\quad(1.2)$$

where now all the denominators depend on $k_0+i\eta$ so that $\Pi_R(K)$ is manifestly holomorphic in the upper-half of the complex k_0 plane. Each ψ_A is a sum or difference of the three particle energies:

$$\psi_A=\pm E_1\pm E_2\pm E_3,\quad(1.3)$$

where the eight combinations of the $\pm$ signs account for the eight different ψ's. The physical interpretation of all these possibilities is standard [11]: The energies that appear in Eq. (1.3) with a positive sign correspond to particles emitted into the heat bath; the energies that appear in with a negative sign correspond to particles absorbed from the heat bath.

Approximate dispersion relations that are real

Using a conventional propagator which has poles at $p_0=\pm(p^2+m^2)^{1/2}$ may not be a good procedure because thermal corrections will shift the pole to a different location. The shift in the pole location is most dramatic in massless theories. For example, in massless $g^2\phi^4/4!$ the one-loop correction will shift the pole to $p_0=\pm(p^2+g^2T^2/24)^{1/2}$. In massless QED and QCD the one-loop corrections to the fermion and gauge-boson self-energies shift the locations of the poles to $p_0=\pm E(p)$, where $E(p)$ is a complicated, real transcendental function of momentum. The retarded propagator for any real dispersion relation is

$$D_R(P)=\frac{1}{(p_0+i\epsilon)^2-E^2(p)}.\quad(1.4)$$

The analysis in this paper will apply to propagators of the form D_R. This propagator is not as complicated as the hard-thermal-loop propagators $^*S(P)$ and $^*D_{\mu\nu}(P)$ for fermion and gauge bosons [1,6,12,13]. The hard-thermal-loop resummed propagators have the structure

$$^*D_R(P)=\frac{1}{(p_0+i\epsilon)^2-p^2-{}^*\Pi_R(p_0,p)}.$$

Although $^*D_R(P)$ has the same poles on the real axis at $p_0=\pm E(p)$ as Eq. (1.4), it also has a branch cut for space-like momentum, $-p\leqslant p_0\leqslant p$. It can be written as

$$^*D_R(P)=\frac{N(p_0,p)}{(p_0+i\epsilon)^2-E^2(p)},$$

where the numerator function N contains the branch cut but has no poles. It seems quite likely that the self-energies computed with $^*D_R$ would contain all the branch points that will be found using D_R and would contain additional branch cuts directly related to the branch cut in N. However, this diversion will not be pursued.

Generalization to complex dispersion relations

From physical considerations one knows that particle propagation is damped at finite temperature. One-loop calculations are misleading in that the solution to $E^2=p^2+m^2+\Pi_R^{1\ \mathrm{loop}}(E,p)$ is always a real energy $E(p)$. If one calculates the self-energy to two-loop accuracy then the pole in the propagator to two-loop accuracy will be complex. The location of this complex pole in the retarded propagator will be denoted by $\mathcal{E}(p)$:

$$\mathcal{E}(p)=\omega(p)-\frac{i}{2}\gamma(p),$$

where ω and γ both real and both positive. The corresponding propagator in the pole approximation is

$$D_R(P)=\frac{1}{[p_0-\mathcal{E}(p)][p_0+\mathcal{E}^*(p)]}.\quad(1.5)$$

It has no singularities for $\mathrm{Im}(p_0)>0$. When p_0 is complex but p is real, it satisfies the condition $D_R(p_0,p)=[D_R(-p_0^*,p)]^*$.

It is important to emphasize that the damping function $\gamma(p)$ cannot just be invented for phenomenological purposes, because when the three-momentum is allowed to become complex, $p_c=p+ip'$, then $\mathcal{E}(p_c)$ must be an analytic function of the variable p_c:

$$\mathcal{E}(p+ip')=\omega(p,p')-\frac{i}{2}\gamma(p,p').\quad(1.6)$$

In particular ω and γ must satisfy the Cauchy-Riemann conditions:

$$\frac{\partial\omega(p,p')}{\partial p}=-\frac{1}{2}\frac{\partial\gamma(p,p')}{\partial p'}$$

$$\frac{\partial\omega(p,p')}{\partial p'}=\frac{1}{2}\frac{\partial\gamma(p,p')}{\partial p}.$$

This guarantees that, in computing a self-energy correction, the integration contours for the momentum variables can be distorted into the complex plane. Without this property the locus of non-analyticity for self-energy corrections would be continuous lines rather than isolated branch points.

Form of the self-energy

The following analysis will employ propagators of the general form in Eq. (1.5). Obviously Eq. (1.4) can be considered as a special case. A particular multi-loop self-energy diagram will have many branch cuts and can be written as the sum of various integrals, each of which displays a unique branch cut [14]. The branch cut of the most general integral results from an intermediate state in which n particles are emitted into the heat bath and n' particles are absorbed from the heat bath and is of the form

$$\Pi_R(K)=\int \prod_{j=1}^{n-1} d^3p_j \prod_{l=1}^{n'} d^3q_l \frac{f(\vec{p},\vec{q})}{k_0-\psi}. \tag{1.7}$$

This definition applies in the region $\mathrm{Im}(k_0)>0$, where the retarded self-energy is holomorphic. Branch point will be sought by analytically continuing away from this region. A particular Feynman diagram will be the sum of several integrals of this form, involving different values of n and n' and often different values of the sum $n+n'$.

Although there are $n+n'$ momenta, one of them is determined by momentum conservation:

$$\vec{k}=\sum_{j=1}^{n} \vec{p}_j+\sum_{l=1}^{n'} \vec{q}_l. \tag{1.8}$$

The denominator function ψ sums over the energies of the emitted particles positively and over the energies of the absorbed particles negatively:

$$\psi=\sum_{j=1}^{n} \mathcal{E}(\vec{p}_j)-\sum_{l=1}^{n'} \mathcal{E}^*(\vec{q}_l). \tag{1.9}$$

The complex energy $\mathcal{E}$ has a negative imaginary part and so ψ automatically has a negative imaginary part. This guarantees that $\Pi_R(K)$ in Eq. (1.7) is holomorphic for $\mathrm{Im}(k_0)>0$.

The range of the function ψ depends on the values of n and n'. For $n\geqslant 2$ and $n'\geqslant 2$ then $-\infty\leqslant\psi\leqslant\infty$ and so the self-energy will have a branch cut along the entire length of the real k_0 axis. For $n\geqslant 2$ and $n'=1$, momentum conservation forces ψ to be bounded from below but not from above. For $n=1$ and $n'\geqslant 2$ momentum conservation forces ψ to be bounded from above but not from below. For $n=1$ and $n'=1$ the range of ψ will be finite.

Most points at which $k_0=\psi$ will not produce singularities in Π_R because the integration contours can generally be distorted so that the integration does not pass over the singularity [provided that $\mathcal{E}(p)$ is analytic in p]. There are two situations which do produce singularities [15,16]. The first, called pinch singularities, occurs at values of k_0 at which two or more singularities of the integrand pinch the integration contour from opposite sides. The necessary condition for a pinch is the simultaneous vanishing of $k_0-\psi$ and of the derivatives of ψ with respect to the $\vec{p}$'s and $\vec{q}$'s. The sufficient conditions require more detailed study of the integrand. The second, called end point singularities, occurs at values of k_0 at which the singularities of the integrand occur at endpoints of the integration region, in this case from $\vec{p}$ and $\vec{q}$ taking on values $\pm\infty$.

It is perhaps worth emphasizing that it is the location of branch points that is under investigation and not the value of the discontinuity across the branch cut. For the present purposes it does not matter if the discontinuity can be grouped as a product of factors from one side of the cut or the other [17].

Applicability to QCD

In QCD the quark dispersion relations are different from the gluon dispersion relations at the one-loop level and certainly at higher loops. The analysis presented here applies to any self-energy contribution in which all of the cut propagators have the same dispersion relation. Thus, for quark self-energies it applies to cuts across intermediate states with all quarks but no gluons. Similarly, for gluon self-energies it applies to cuts across intermediate states that are composed entirely of gluons or entirely of quarks. The inability to treat intermediate states with mixtures of particle species is obviously a limitation and it will require more work to overcome. Even at $T=0$, unequal masses are difficult to treat.

Sample result

A simple but interesting example of the results that will be derived occurs for self-energy diagrams with a three-particle intermediate state. Three-particle intermediate states in which all three particles are the same species occur at two-loop order in the following cases: (a) scalar field self-energy with ϕ^4 interaction as already discussed; (b) gluon self-energy in QCD with three-gluon intermediate state; (c) quark self-energy in QCD with a three-quark intermediate state[18]. In all these examples $n+n'=3$. One contribution allows two particles to be emitted into the heat bath and one particle to be absorbed from the heat bath $(n=2,n'=1)$. Subsequent analysis will show that there will be three branch points: viz. at $k_0=\pm k$ and at $k_0=\infty$ with branch cuts connecting them. The branch points at $k_0=\pm k$ will be essential singularities with behavior $\exp[3m^2k/(K^2T)]$ as K^2 approaches zero from the negative region. Here m is the effective thermal mass from the large-momentum expansion of the dispersion relation.

If the single-particle energy $\mathcal{E}(p)$ used to define the loop expansion is complex, then only the above three branch points occur. If, however, a real energy $E(p)$ is used then the self-energy will have a fourth branch point at $k_0=E(k)$. In this situation the propagator which was assumed to have a simple pole at $k_0=E(k)$ turns out to generate a branch point also at $k_0=E(k)$. This ugliness infects any perturbative expansion built on a real dispersion relation.

Organization

It is assumed throughout that the branch cuts of thermal self-energies come entirely from the denominators of the propagator functions and are not affected by the spin of the particles.

Section II presents three toy examples of functions with branch cuts that extend from $-\infty$ to $+\infty$, as this does not occur for zero-temperature self-energies. Two of these examples have additional branch points on the real axis that illustrate features that will be found in the actual self-energy.

Section III analyzes the branch points that occur for the general intermediate state consisting of n particles emitted into the heat bath and n' absorbed from the heat bath. Section III D summarizes the results and may be read independently of the development sections.

Section IV discusses some implications of the results.

There are four Appendixes. Appendixes A and B contain detailed proofs that complete the arguments given in Sec. III. Appendix C is an explicit one-loop example that displays the essential singularity at $k_0=\pm k$. Appendix D is an explicit two-loop example from Wang and Heinz [19] that shows both the essential singularity at the light cone and the branch point at the mass shell.

II. SIMPLE FUNCTIONS WITH BRANCH POINTS AT $\pm\infty$

One of the main results of this paper will be that self-energies at $T\neq 0$ not only have branch points at the ends of their branch cuts but also have extra branch points not at the ends but attached to the cuts. Although this is unfamiliar from $T=0$ physics, it is not very exotic mathematics. This section contains three toy examples involving one-dimensional integrals that can be computed exactly.

Each example concerns a function defined by an integral of the form

$$F(\omega)=\int_{-\infty}^{\infty} dz \frac{1}{\omega-\psi(z)}, \tag{2.1}$$

where $\psi(z)$ is a real function when z is real. When $\mathrm{Im}\,\omega>0$ the function is holomorphic and defines the retarded form of $F(\omega)$. When $\mathrm{Im}\,\omega<0$ the function is holomorphic and defines the advanced form of $F(\omega)$.

In the following examples the function $\psi(z)$ will be real and chosen so that $F(\omega)$ have a branch cut running from $\omega=-\infty$ to $\omega=\infty$, which separates the two regions of holomorphicity. Such a cut requires that $\psi(z)$ takes on all real values.

The branch points in the three examples can be found by examining the integrands and are confirmed by explicit integration. The discontinuity across the branch cut is pure imaginary $F(\omega_r+i\epsilon)-F(\omega_r-i\epsilon)=2i\,\mathrm{Im}\,F(\omega_r)$, where

$$\mathrm{Im}\,F(\omega_r)=-\pi\int_{-\infty}^{\infty} dz\,\delta[\omega_r-\psi(z)]. \tag{2.2}$$

The discontinuity formula is not the best way to answer the question of whether $F(\omega)$ has any branch points at finite real values of ω that are attached to the branch cut.

Example 1. The first example is

$$f(\omega)=\int_{-\infty}^{\infty} dz \frac{1}{\omega-\sinh z}. \tag{2.3}$$

For any real value of ω, positive or negative, there is a real value of z at which the denominator of the integrand vanishes and this leads to a branch cut along the entire real axis. The end points $z=\pm\infty$ of the integration produce the branch points at $\omega=\pm\infty$. Explicit integration confirms this:

$$f(\omega)=\frac{1}{\sqrt{\omega^2+1}}\ln\left[\frac{\omega+\sqrt{\omega^2+1}}{\omega-\sqrt{\omega^2+1}}\right]. \tag{2.4}$$

There are branch points at $\omega=\pm\infty$, where the argument of the logarithm vanishes. Inspection shows that Eq. (2.4) is discontinuous across the real axis. If ω approaches the real axis from above, then the argument of the logarithm approaches $e^{-i\pi}|R|$; if ω approaches the real axis from below, then the argument of the logarithm approaches $e^{i\pi}|R|$. In this example there are no branch points at finite values of ω. This is the type of behavior that is usually thought to be typical of finite temperature field theory.

Example 2. Next consider

$$g(\omega)=\int_{-\infty}^{\infty} dz \frac{1}{\omega-z^3}. \tag{2.5}$$

This integral has end-point singularities at $\omega=\pm\infty$ and also a pinch singularity at $\omega=0$. The pinch occurs because at $z=0$ both $\psi(z)$ and $d\psi/dz$ vanish [15,16].

The integral may easily be evaluated by Cauchy's theorem. For any ω the integrand contains three simple poles as a function of z. When ω is in the upper half-plane, there are two poles in z above the real axis and one pole below. Integration gives

$$\mathrm{Im}\,\omega>0: \quad g(\omega)=\frac{2\pi i}{3}\frac{e^{-i2\pi/3}}{\omega^{2/3}}. \tag{2.6}$$

When ω is in the lower half-plane, then $g(\omega)$ is the complex conjugate of the above:

$$\mathrm{Im}\,\omega<0: \quad g(\omega)=\frac{-2\pi i}{3}\frac{e^{i2\pi/3}}{\omega^{2/3}}. \tag{2.7}$$

As expected, $g(\omega)$ has a branch cut along the full length of the real axis with branch points at $\omega=\pm\infty$. The new feature is the third branch point at $\omega=0$.

It is useful to investigate the analytic structure a bit more. Let ω_0 lie in the upper half-plane, where $g(\omega)$ given by Eq. (2.6) is analytic. To explore $g(\omega)$ in the neighborhood of ω_0, set $\omega=\omega_0+re^{i\phi}$ with r real. As ϕ increases from 0 to 2π, ω moves in a circle of radius r centered on ω_0. This circle can pass into the lower half-plane since Eq. (2.6) can be analytically continued into the lower half-plane. If $r<|\omega_0|$ the the circle will not pass around the origin and the function $(\omega_0+re^{i\phi})^{2/3}$ will have the same value at $\phi=0$ and at

$\phi=2\pi$. However, if $r>|\omega_0|$ then ω will encircle the origin and $g(\omega)$ will not return to its original value. To clarify this, choose $\omega_0=0$ so that $\omega=re^{i\phi}$. Then when ϕ increases from 0 to 2π, $\omega^{2/3}$ will return to the value $e^{4\pi i/3}\omega^{2/3}$ and $g(\omega)$ will return to the value

$$g_{II}(\omega)=\frac{2\pi i}{3}\frac{1}{\omega^{2/3}}. \tag{2.8}$$

This shows that the function $g(\omega)$ has a branch point at $\omega=0$ in addition to those at $\omega=\pm\infty$. [If ω encircles the origin two more times in a counterclockwise direction then $g(\omega)$ will return to the original value in Eq. (2.6).]

Example 3. The third example is

$$h(\omega)=\int_{-\infty}^{\infty}dz\frac{1}{2\omega-z^3+3z}. \tag{2.9}$$

In addition to end-point singularities at $\omega=\pm\infty$, this integral has pinch singularities at $\omega=\pm1$. The pinch singularities arise because $\psi(z)=z^3-3z$ has a local maximum at $z=1$ and a local minimum at $z=-1$. Consequently the denominator has a double zero at $\omega=\pm1$ [15,16].

Since the integrand has three simple poles, the integral can be performed using Cauchy's theorem. For $\mathrm{Im}\,\omega>0$ the result is

$$\mathrm{Im}\,\omega>0:\quad h(\omega)=\frac{2\pi i}{3}\frac{1}{e^{2\pi i/3}A^{2/3}+e^{-2\pi i/3}B^{2/3}+1}, \tag{2.10}$$

where

$$A=\omega+\sqrt{\omega^2-1},\quad B=\omega-\sqrt{\omega^2-1}.$$

As expected, $h(\omega)$ has four branch points on the real axis: at $\omega=-\infty$ where A vanishes; at $\omega=\infty$, where B vanishes; and at $\omega=\pm1$, where A and B have branch points.

The discontinuity of $h(\omega)$ across the real axis can be computed either directly from Eq. (2.10) or by using Eq. (2.2). By either method the result is

$$\omega^2>1:\quad \mathrm{Im}\,h(\omega)=-\frac{\pi}{3}\frac{1}{A^2+B^2+1} \tag{2.11}$$

$$\omega^2<1:\quad \mathrm{Im}\,h(\omega)=\frac{2\pi}{3}\frac{1}{2\cos[(2\theta+2\pi)/3]+1}, \tag{2.12}$$

where for $-1<\omega<1$ the angle θ is defined by $\omega=\cos\theta$. The imaginary part has a different value as ω approaches 1 from above or from below. For infinitesimal ϵ,

$$\omega=1+\epsilon:\quad \mathrm{Im}\,h(1+\epsilon)=-\frac{\pi}{9} \tag{2.13}$$

$$\omega=1-\epsilon:\quad \mathrm{Im}\,h(1-\epsilon)=-\infty. \tag{2.14}$$

Thus the imaginary part of $h(\omega)$ is discontinuous at the branch point. This method will be used in Appendix D.

III. BRANCH POINTS OF SELF-ENERGIES

This section will examine the general problem of a perturbative expansion based on propagators of the form Eq. (1.5) in which $\mathcal{E}(p)$ is any single-particle energy, real or complex. A summary of this section is given in Sec. III D.

A. Branch points for *n* emissions with no absorptions

The simplest type of branch cuts are those that come from the production of n particles. After integrations over the time-like components of the loop momenta, the retarded thermal, self-energy can be written as an integral over $n-1$ independent three momenta:

$$\Pi_R(k_0,k)=\int\frac{d^3p_1d^3p_2\ldots d^3p_{n-1}f(\vec{p}_j)}{k_0-\psi}. \tag{3.1}$$

The numerator $f(\vec{p}_j)$ will depend on temperature and and on the spins of the particles. The denominator function ψ is

$$\psi=\sum_{j=1}^{n}\mathcal{E}(\vec{p}_j).$$

The momentum of the last particle, viz. $\vec{p}_n$, is determined by momentum conservation

$$\vec{k}=\sum_{j=1}^{n}\vec{p}_j.$$

A value of k_0 that makes the denominator of the integrand in Eq. (3.1) vanish will rarely produce a singularity in Π_R because the integration contour can be distorted into the complex plane so as to avoid the point at which the denominator vanishes. Another way to describe this situation is to focus on the values of $\vec{p}_j$ that make the denominator vanish for a particular k_0. As k_0 varies, the location of the critical $\vec{p}_j$ varies. At a particular k_0 the singularity may move onto the real $\vec{p}_j$ axis. This will generally not produce a singularity of the function Π_R because the contour can be distorted so as to avoid the singularity [15,16]. However, if two singularities move so as to pinch the contour between them at a particular k_0 then the function Π_R will have a singularity at that k_0. The necessary condition for the denominator of Eq. (3.1) to have a double pole at some particular k_0 requires that both the denominator and its first derivative vanish [15,16].

It is convenient to implement momentum conservation by employing a Lagrange multiplier $\vec{v}$ and defining a new function Ψ as

$$\Psi=\sum_{j=1}^{n}\mathcal{E}(\vec{p}_j)+\vec{v}\cdot\left(\vec{k}-\sum_{j=1}^{n}\vec{p}_j\right). \tag{3.2}$$

Any point at which the derivatives of Ψ with respect to $\vec{p}_1,\ldots,\vec{p}_n$ and $\vec{v}$ all vanish will be a point at which the

derivatives of ψ with respect to $\vec{p}_1, \ldots, \vec{p}_{n-1}$ vanish while keeping momentum conserved.

To proceed further it is helpful to introduce the group velocity

$$V(p) = \frac{d\mathcal{E}(p)}{dp}, \tag{3.3}$$

which may be complex when $\mathcal{E}$ is complex. The pinch conditions

$$0 = \frac{\partial \Psi}{\partial \vec{p}_j} = \hat{p}_j V(p_j) - \vec{v},$$

imply that all the $\vec{p}_j$ are equal. The common value of $\vec{p}_j$ is determined by extremizing with respect to the Lagrange multiplier:

$$0 = \frac{\partial \Psi}{\partial \vec{v}} = \vec{k} - \sum_{j=1}^{n} \vec{p}_j,$$

and this fixes $\vec{p}_j = \vec{k}/N$. The extreme value of ψ is

$$\psi_{\rm ext} = n\mathcal{E}(\vec{k}/n). \tag{3.4}$$

Thus there will be a branch point at $k_0 = n\mathcal{E}(\vec{k}/n)$. For the free-particle dispersion relation the branch point is at $k_0 = \sqrt{k^2 + (nm)^2}$. When $\mathcal{E}$ is an effective thermal energy, the branch point at $k_0 = nE(k/n)$ will be temperature dependent. In either case, the branch cut runs parallel to the positive k_0 axis and terminates with a branch point at $k_0 = \infty$.

For the related situation of n particles absorbed from the thermal bath, then $\psi = -\sum_{j=1}^{n} \mathcal{E}^*(\vec{p}_j)$. This produces a branch point at $k_0 = -n\mathcal{E}^*(\vec{k}/n)$ and a branch cut which runs parallel to the negative k_0 axis and terminates at $-\infty$.

B. Branch points that only occur for real group velocities

The part of the self-energy which has an intermediate state consisting of n emitted particles and n' absorbed particles has the form

$$\Pi_R(K) = \int \prod_{j=i}^{n-1} d^3 p_j \prod_{l=1}^{n'} d^3 q_l \frac{f(\vec{p}, \vec{q})}{k_0 - \psi} \tag{3.5}$$

where the momentum $\vec{p}_n$ is determined by momentum conservation, Eq. (1.8), and ψ is given by Eq. (1.9).

To examine for pinch singularities in momentum space subject to the constraint of momentum conservation, it is again convenient to introduce a Lagrange multiplier $\vec{v}$ and define a new function

$$\Psi = \sum_{j=1}^{n} \mathcal{E}(\vec{p}_j) - \sum_{l=1}^{n'} \mathcal{E}^*(\vec{q}_l) + \vec{v} \cdot \left(\vec{k} - \sum_{j=1}^{n} \vec{p}_j - \sum_{l=1}^{n'} \vec{q}_l \right). \tag{3.6}$$

The pinch condition

$$0 = \frac{\partial \Psi}{\partial \vec{p}_j} = \hat{p}_j V(p_j) - \vec{v}$$

implies that all $\vec{p}_j$ are equal. The condition

$$0 = \frac{\partial \Psi}{\partial \vec{q}_l} = -\hat{q}_l V^*(q_l) - \vec{v}$$

implies that all the $\vec{q}_l$ are equal. Eliminating the Lagrange multiplier $\vec{v}$ in these last two conditions gives

$$\hat{p}_j V(p_j) = -\hat{q}_l V^*(q_l). \tag{3.7}$$

The third condition is

$$0 = \frac{\partial \Psi}{\partial \vec{v}} = \vec{k} - \sum_{j=1}^{n} \vec{p}_j - \sum_{l=1}^{n'} \vec{q}_l. \tag{3.8}$$

Case 1. $\mathcal{E}$ real: When the single particle energy $\mathcal{E}$ is real, it is denoted by E. The group velocity V is real. Equation (3.7) implies first that $\hat{p}_j = -\hat{q}_l$ and second that $V(p_j) = V(q_l)$. This is solved by $\vec{p}_j = -\vec{q}_l$. Equation (3.8) can then be solved for $n \neq n'$:

$$\vec{p}_j = \frac{\vec{k}}{n - n'}; \quad \vec{q}_l = \frac{-\vec{k}}{n - n'}.$$

(For $n = n'$ there is no solution.) The value of Eq. (3.6) at the extremum is

$$\psi_{\rm ext} = [n - n'] E(k/[n - n']). \tag{3.9}$$

The necessary conditions for a branch point at $k_0 = \psi_{\rm ext}$ are thus satisfied. Because this extrememum is a saddle point and not a local maximum or minimum, the conventional experience does not apply. To demonstrate that there actually is a pinch of the integration contour requires more analysis. This analysis is done in Appendix A and confirms that there is a branch point at $k_0 = \psi_{\rm ext}$ and also shows that the branch point has infinitely many sheets.

For the free particle dispersion relation the branch point is at $k_0 = \sqrt{k^2 + [(n - n')m]^2}$. If E is a temperature-dependent effective energy, then the location of the branch cut will be temperature dependent.

The most surprising consequence of this is that when $n - n' = 1$, there is a branch point at $k_0 = E(k)$, which is precisely at the location of the pole in the propagator that was used to define the perturbative series. An example of this phenomena occurs in the self-energy in ϕ^4 theory. Wang and Heinz [19] have calculated the imaginary part of the two-loop self-energy. Appendix D shows explicitly that the two-loop self-energy has a branch point at the mass-shell.

When $n - n' = 2$ there is a branch point at $k_0 = 2E(\vec{k}/2)$ that occurs by cutting $n + n'$ propagators. For $n' = 0$ this is the two-particle normal threshold already displayed in Eq. (3.4). But for $n' \neq 0$ the branch point occurs in more com-

plicated diagrams than in Eq. (3.4). Similarly, for $n-n'=7$ the branch point at $k_0=7E(\vec{k}/7)$ occurs in diagrams with $n+n'\geqslant 7$.

Case 2. $\mathcal{E}$ complex but V real: It is possible to have $\mathcal{E}(p)=E(p)-i\gamma/2$ but γ is a non-zero constant. The true damping cannot be constant, but the constant γ approximation is sometimes useful [20]. The group velocity $V=dE/dp$ is real so that the pinch condition is satisfied at the same momenta $\vec{p}_j$ and $\vec{q}_l$ as in case 1. The only difference is that the extremum of Eq. (3.6) is

$$\psi_{\rm ext}=[n-n']E(k/[n-n'])-i(n+n')\frac{\gamma}{2}. \quad (3.10)$$

When $n-n'=1$ the various branch points at $k_0=E(k)-i(n+n')\gamma/2$ do not coincide with the single-particle pole at $k_0=E(k)-i\gamma/2$. The proof in Appendix A includes this case.

Case 3. $\mathcal{E}$ and V complex: When the single-particle energy $\mathcal{E}$ is complex, it is difficult to solve Eqs. (3.7) and (3.8). The first equation implies that $\hat{p}_j=\pm\hat{q}_l$. Let us examine the case $\hat{p}_j=-\hat{q}_l$. Then

$$V(p_j)=V^*(q_l).$$

It is possible to invent an analytic function $V(p)$ that satisfies this condition at special momentum. However, the branch points would then be artifacts of the approximation scheme. For the exact value of the single-particle pole energy, $\mathcal{E}_{\rm pole}$, the imaginary part is negative and vanishes at zero momentum and at infinite momentum. Therefore its first derivative must be negative at small momentum and positive at large momentum. If p_j is small and q_l is large, it may be possible for $\mathrm{Im}\,V(p_j)=-\mathrm{Im}\,V(q_l)$. Whether the real parts would satisfy $\mathrm{Re}\,V(p_j)=\mathrm{Re}V(q_l)$ seems unlikely.

C. Essential singularities at $k_0=\pm k$

The branch points discussed above occur when $\vec{p}_j$ and $\vec{q}_l$ all have a finite magnitude. Additional branch points can result from pinches at infinite values of $\vec{p}_j$ and $\vec{q}_l$. To investigate these it is necessary to make some assumption about the behavior of the dispersion relation $\mathcal{E}(p)$ at large momenta. It will be assumed that

$$p\to\infty: \quad \mathcal{E}(p)\to p+\frac{m^2}{2p}+\cdots \quad (3.11)$$

and that the imaginary part of $\mathcal{E}(p)$ falls faster than $1/p$. This is obviously the correct asymptotic behavior for any theory that is massive at zero temperature. Theories that are massless at zero temperature require a resummation to obtain a sensible dispersion relation $\mathcal{E}(p)$. In this case the parameter m plays acts as an effective thermal mass at large momentum. The asymptotic behavior of one-loop dispersion relations in massless gauge theories is well known [6,12,13]. The asymptotic behavior Eq. (3.11) applies to spinless fields, to the spinor field components which have the same helicity as chirality, and to the vector field components that are transversely polarized. It does not apply to the spinor field components that have the helicity opposite to the chirality nor to the longitudinally polarized vector bosons. [Both these cases have asymptotic behavior $\mathcal{E}(p)\to p+Ap\,\exp(-p^2/m^2)$. However, in these two cases the residue of the pole vanishes at large momentum like $\exp(-p^2/m^2)$.]

To investigate the branch points that can occur at large momenta, it is useful to introduce three Lagrange multiplier vectors: $\vec{v}_1,\vec{v}_2,\vec{P}$ and define

$$\Psi=\sum_{j=1}^{n}\mathcal{E}(\vec{p}_j)-\sum_{l=1}^{n'}\mathcal{E}^*(\vec{q}_l)+\vec{v}_1\cdot\left(\frac{1}{2}\vec{k}+\vec{P}-\sum_{j=1}^{n}\vec{p}_j\right)+\vec{v}_2\cdot\left(\frac{1}{2}\vec{k}-\vec{P}-\sum_{l=1}^{n'}\vec{q}_l\right). \quad (3.12)$$

This is equivalent to Eq. (3.6) because extremizing with respect to $\vec{P}$ sets $\vec{v}_1=\vec{v}_2$. However, we will compute the extrema of Eq. (3.12) by computing the derivatives in a different order. The pair of conditions

$$0=\frac{\partial\Psi}{\partial\vec{p}_j}=V(p_j)\hat{p}_j-\vec{v}_1$$

$$0=\frac{\partial\Psi}{\partial\vec{v}_1}=\frac{1}{2}\vec{k}+\vec{P}-\sum_{j=1}^{n}\vec{p}_j,$$

imply that all $\vec{p}_j$ are equal and that they have the common value

$$\vec{p}_j=\left(\frac{1}{2}\vec{k}+\vec{P}\right)/n. \quad (3.13)$$

The pair of conditions

$$0=\frac{\partial\Psi}{\partial\vec{q}_l}=-V^*(q_l)\hat{q}_l-\vec{v}_2$$

$$0=\frac{\partial\Psi}{\partial\vec{v}_2}=\frac{1}{2}\vec{k}-\vec{P}-\sum_{l=1}^{n'}\vec{q}_l$$

imply that all the $\vec{q}_l$ are equal and have the common value

$$\vec{q}_l=\left(\frac{1}{2}\vec{k}-\vec{P}\right)/n'. \quad (3.14)$$

As a result,

$$\Psi=n\mathcal{E}\left(\frac{1}{n}|\vec{P}+\vec{k}/2|\right)-n'\mathcal{E}^*\left(\frac{1}{n'}|\vec{P}-\vec{k}/2|\right).$$

The condition $0=\partial\Psi/\partial\vec{P}$ requires

$$\frac{\vec{P}+\vec{k}/2}{|\vec{P}+\vec{k}/2|}V\left(\frac{1}{n}\left|\vec{P}+\frac{\vec{k}}{2}\right|\right)=\frac{\vec{P}-\vec{k}/2}{|\vec{P}-\vec{k}/2|}V^*\left(\frac{1}{n'}\left|\vec{P}-\frac{\vec{k}}{2}\right|\right).$$

Regardless of the value of the group velocity in this equation, the two vectors $\vec{P}+\vec{k}/2$ and $\vec{P}-\vec{k}/2$ can only be proportional to each when $\vec{P}_\perp=0$, where $\vec{P}=\hat{k}P_\parallel+\vec{P}_\perp$. When $\vec{P}_\perp=0$ the vectors multiplying V and V^*, respectively, both are equal to the unit vector $\hat{k}$. Thus the condition reduces to

$$V\left(\frac{1}{n}\left|P_\parallel+\frac{k}{2}\right|\right)=V^*\left(\frac{1}{n'}\left|P_\parallel-\frac{k}{2}\right|\right). \tag{3.15}$$

Because of the presumed asymptotic behavior in Eq. (3.11), this is satisfied in the limit $P_\parallel\to\pm\infty$. If V is complex, this is the only possible solution. If V is real, then in addition to the solution for infinite $P_\parallel$ there is also a finite solution when $n\neq n'$, namely $\vec{P}_\parallel+k/2=nk/(n-n')$. The finite solution was already treated in Sec. III B and requires no further discussion.

Thus, regardless of the values of n and n', at $P_\parallel\to\infty$ the necessary conditions are satisfied for a branch point. From Eqs. (3.13) and (3.14) and the fact that $\vec{P}_\perp=0$, the important region of integration is

$$\vec{p}_j=\frac{\hat{k}}{n}\left(\frac{k}{2}+P_\parallel\right),\quad \vec{q}_l=\frac{\hat{k}}{n}\left(\frac{k}{2}-P_\parallel\right). \tag{3.16}$$

The denominator function is

$$\psi=n\mathcal{E}\left(\frac{1}{n}\left|P_\parallel+\frac{k}{2}\right|\right)-n'\mathcal{E}^*\left(\frac{1}{n'}\left|P_\parallel-\frac{k}{2}\right|\right).$$

The asymptotic behavior assumed in Eq. (3.11) implies

$$P_\parallel\to+\infty:\ \psi\to k+\frac{(nm)^2}{2P_\parallel+k}-\frac{(n'm)^2}{2P_\parallel-k}+\cdots. \tag{3.17}$$

Therefore the branch point in $\Pi_R(K)$ produced by the denominator $k_0-\psi$ will occur at $k_0=k$. The region $P_\parallel\to-\infty$ produces a branch point at $k_0=-k$. As in Sec. III B the arguments thus far presented are only necessary conditions for a branch point. To show sufficiency requires a more detailed analysis and this is provided in Appendix B.

Why an essential singularity

In Secs. III A and III B the branch points were produced by particle momenta that were finite. Here the branch points at $k_0=\pm k$ are produced by momenta that are infinite and this makes it possible to show that the branch points are essential singularities.

The effect comes from the statistical factor, S, in the integrand of the self-energy contribution that contains n emitted particles and n' absorbed particles:

$$S=S_{\rm direct}-\sigma S_{\rm inverse}, \tag{3.18}$$

where $\sigma=1$ for a boson self-energy and $\sigma=-1$ for a fermion self-energy. The statistical factors are

$$S_{\rm direct}=\prod_{j=1}^{n}\,[1+\sigma_j N_j]\prod_{l=1}^{n'}\,N_l^* \tag{3.19}$$

$$S_{\rm inverse}=\prod_{j=1}^{n}\,N_j\prod_{l=1}^{n'}\,[1+\sigma_l N_l^*] \tag{3.20}$$

where $\sigma=\pm1$ for bosons and fermions and

$$N_j=1/[\exp(\beta\mathcal{E}(\vec{p}_j))-\sigma_j]$$

$$N_l^*=1/[\exp(\beta\mathcal{E}^*(\vec{q}_l))-\sigma_l].$$

Because of Eqs. (3.11) and (3.16), in the region $P_\parallel\to\infty$ the statistical factor becomes

$$S\to(e^{\beta k/2}-\sigma e^{-\beta k/2})\exp(-\beta P_\parallel). \tag{3.21}$$

By finding the way in which $P_\parallel$ approaches infinity as $k_0\to k$, one can be more specific about the nature of the branch point. Near the branch point, k_0-k is very small but non-zero, and the self-energy denominator $k_0-\psi$ will vanish when $P_\parallel$ is very large but not actually infinite. The condition $k_0-\psi=0$ gives a quadratic equation for $P_\parallel$. The two roots are

$$P_{\parallel\pm}=\frac{1}{4(k_0-k)}((n^2-n'^2)m^2\pm\{[2k(k_0-k)-(n^2+n'^2)m^2]^2-[2nn'm^2]^2\}^{1/2}).$$

There are two cases to be distinguished.

Case 1. $n=n'$: If k_0 is real and approaches k from below, the root that approaches $+\infty$ is

$$P_{\parallel-}=\frac{k}{2}\sqrt{1-\frac{2(nm)^2}{k(k_0-k)}}. \tag{3.22}$$

The statistical factor Eq. (3.21) becomes

$$S\to(e^{\beta k/2}-\sigma e^{-\beta k/2})\exp\left(-\frac{\beta k}{2}\sqrt{1-\frac{2(nm)^2}{k(k_0-k)}}\right). \tag{3.23}$$

This is an essential singularity at $k_0=k$. If k_0 is real and approaches k from below, then $S\to0$. More generally, the behavior depends on how k_0-k approaches zero in the complex plane. Appendix C provides a one-loop calculation with $n=n'=1$ that displays this behavior in Eq. (C3).

Case 2. $n\neq n'$: For definiteness take $n>n'$. Then if k_0 is real and approaches k from above, the root that approaches $+\infty$ is

$$P_{\parallel+}\to\frac{(n^2-n'^2)m^2}{2(k_0-k)}.$$

The statistical factor Eq. (3.21) becomes

$$S\to(e^{\beta k/2}-\sigma e^{-\beta k/2})\exp\left(-\frac{\beta(n^2-n'^2)m^2}{2(k_0-k)}\right). \tag{3.24}$$

This, again, is an essential singularity at $k_0=k$, whose behavior naturally depends upon the direction from which k_0 approaches k. Appendix D provides a two-loop calculation with $n=2$, $n'=1$, and $k=0$. The exponent is predicted to be $-\beta 3m^2/2k_0$, and this is just what is found in Eq. (D14).

Comment on hard thermal loops: The one-loop gluon self-energy has branch points at $k_0=\pm k$ [6,12,13]. These come from intermediate states with $n=1,n'=1$ (either two gluons or two fermions). The one-loop calculations are done using a massless dispersion relation for the intermediate particles. Therefore $m=0$ in Eq. (3.11) so that $P_{\parallel+}=k/2$ in Eq. (3.22). Since this momenta is finite, the statistical factor S is unremarkable and cannot produce an essential singularity. Explicit calculations show that the branch points at $k_0=\pm k$ are logarithmic for the hard thermal loops.

D. Summary

In the following summary the single-particle energies $\mathcal{E}(p)$ can be complex or real. For results that only apply when the energies are real, the block E will be used instead of the script $\mathcal{E}$. Real E produces the exceptional branch points discussed in Sec. III B and they will be described in parentheses in the summary below. The ubiquitous branch points at $k_0=\pm k$ are always essential singularities and this will not be repeated each time. As noted following Eq. (1.9), when n and/or n' have the value 1, the range of ψ is constrained by momentum conservation and this often determines the end points of the branch cuts.

1. Organized by n', the number of absorptions

The most concise way to summarize the previous results is by n', the number of particles absorbed from the heat bath.

(*a*) *No absorptions:* $n'=0$. This is the simplest case and directly analogous to zero temperature. The branch cut is semi-infinite: $n\,\mathcal{E}(k/n)\leqslant k_0\leqslant\infty$.

(*b*) *One absorption:* $n'=1$. There are two subcases. If $n=1$ then the branch cut is only for space-like four momenta: $-k\leqslant k_0\leqslant k$. If $n\geqslant 2$ then the branch cut is semi-infinite: $-k\leqslant k_0\leqslant\infty$; and there is a branch point at $k_0=k$. (If the single-particle energies are real, there is an exceptional branch point at $k_0=[n-1]E(k/[n-1])$. For $n=2$ the last branch point coincides with the free particle pole at $k_0=E(k)$.)

(*c*) *Two or more absorptions:* $n'\geqslant 2$. There are three subcases. If $n=0$ the branch cut is semi-infinite: $-\infty\leqslant k_0\leqslant -n'\mathcal{E}^*(k/n')$. If $n=1$ the branch cut is also semi-infnte: $-\infty\leqslant k_0\leqslant k$, and there is an additional branch point at $k_0=-k$. If $n\geqslant 2$ the branch cut runs the full length of the real axis: $-\infty\leqslant k_0\leqslant\infty$; and there are two additional branch points at $k_0=\pm k$. (If the single-particle energies are real, there are exceptional branch points for $n\neq n'$ at $k_0=[n-n']E(k/[n-n'])$. Whenever $n-n'=\pm 1$ this last branch point coincides with the free particle poles at $k_0=\pm E(k)$.)

2. Organized by $n+n'$, the number of particles in the intermediate state

A particular diagram can generally be cut in several possible ways. Each cut is through a particular number of propagators or equivalently through an intermediate state with a particular number of particles. As a practical matter, this is perhaps the most useful way to summarize the cut structure.

(*a*) *Two-particle states:* Two-particle intermediate states, $n+n'=2$, are possible with a cubic coupling but not with a quartic coupling. There are three types of branch cuts. For $n=2,n'=0$ there will be a semi-infinite cut $2\mathcal{E}(k/2)\leqslant k_0\leqslant\infty$. For $n=n'=1$ there will be a finite length branch cut $-k\leqslant k_0\leqslant k$. For $n=0,n'=2$ there will be a semi-infinite branch cut $-\infty\leqslant k_0\leqslant -2\mathcal{E}^*(k/2)$.

(*b*) *Three-particle states:* The possibility $n+n'=3$ occurs with both cubic and quartic coupling. There are four types of branch cuts. For $n=3,n'=0$ there will be a semi-infinite branch cut $3\mathcal{E}(k/3)\leqslant k_0\leqslant\infty$. For $n=2,n'=1$ there will be a semi-infinite branch cut for $-k\leqslant k_0\leqslant\infty$ and in addition there will be a branch point attached to the cut at $k_0=k$. [If the single-particle energies are real, there will be an exceptional branch point at $k_0=E(k)$.] For $n=1$, $n'=2$ the range of the branch cut is $-\infty\leqslant k_0\leqslant k$ with an attached branch point at $k_0=-k$. [If the single-particle energies are real, there will be an exceptional branch point at $k_0=-E(k)$.] For $n=0$, $n'=3$ the extent of the branch cut will be $-\infty\leqslant k_0\leqslant -3\mathcal{E}^*(k/3)$.

(*c*) *Four-particle states:* For a cut through $n+n'=4$ propagators, there are five different types of branch cuts. For $n=4$, $n'=0$ there is only a four-particle production cut for $4\mathcal{E}(k/4)\leqslant k_0\leqslant\infty$. For $n=3$, $n'=1$ the range of the branch cut is $-k\leqslant k_0\leqslant\infty$ with an additional branch point at $k_0=k$. [If the single-particle energies are real, there will be an exceptional branch point at $k_0=2E(k/2)$.] For $n=n'=2$ the branch cut runs the full length of the real axis, $-\infty\leqslant k_0\leqslant\infty$ with two additional branch points at $k_0=-k$ and $k_0=k$. For $n=1$, $n'=3$ the range of the branch cut is $-\infty\leqslant k_0\leqslant k$ with an additional branch point at $k_0=-k$. [If the single-particle energies are real, there will be an exceptional branch point at $k_0=-2E(k/2)$.] For $n=0$, $n'=4$ there is only a four-particle absorption cut for $-\infty\leqslant k_0\leqslant -4\mathcal{E}^*(k/4)$.

IV. COMMENTS

A. Expectations for the exact self-energy

Section III A showed that there will be a branch point in the retarded self-energy $\Pi_R(k_0,\vec{k})$ at $k_0=n\mathcal{E}(\vec{k}/n)$ that results from the emission of n particles and, likewise, a branch point at $k_0=-n\mathcal{E}^*(\vec{k}/n)$ that results from the absorption of n particles from the heat bath. The effective single-particle energies will generally be temperature dependent and complex and so the location of the branch points will generally be temperature dependent and complex. Furthermore, the location of the branch points is model dependent in the sense that one can change the single-particle energies $\mathcal{E}$ and thus change the location of the branch points in the perturbative expansion.

However, the location of the branch points in the exact self-energy cannot be model dependent. If one summed the perturbative self-energy contributions to all orders, the model-dependence of the branch points would disappear just

as the model-dependence of the propagator pole would disappear. If the exact propagator has a pole at $\mathcal{E}_{\rm exact}(p)$, the exact self-energy should have normal-threshold branch points at $k_0=n\mathcal{E}_{\rm exact}(\vec{k}/n)$ and at $k_0=-n\mathcal{E}^*_{\rm exact}(\vec{k}/n)$. In addition, there will be essential singularities at $k_0=\pm k$.

B. Bad features of real dispersion relations

Performing perturbative calculations using the free thermal propagator or indeed any thermal propagator containing a real dispersion relation $E(p)$ leads to a self-energy that has branch points at the perturbative mass shell, $k_0=E(k)$. If $\Pi_R(k_0)$ is the retarded self-energy computed beyond one-loop order using a real energy $E(p)$ to define the perturbation series, then to improve on the value of $E(p)$ one needs to solve perturbatively for $\mathcal{E}_{\rm pole}$

$$\mathcal{E}_{\rm pole}=\sqrt{E^2+\Pi_R(\mathcal{E}_{\rm pole})}. \tag{4.1}$$

A perturbative expansion means that $\Pi_R(E)$ and its derivatives at E are small compared to E. Thus the lowest-order contribution to the damping rate should be

$$\mathrm{Im}\,\mathcal{E}_{\rm pole}\approx \mathrm{Im}[\Pi_R(E)]/2E. \tag{4.2}$$

However, this will fail at two-loop order because $\Pi_R(k_0)$ has a branch point precisely at $k_0=E$. This was first encountered in calculations of the fermion damping rate in QCD, where it was found that even with a magnetic mass to eliminate the infrared divergence there is a branch point at $k_0=E$ that comes from the intermediate state with $n=2,n'=1$ [18]. Appendix D contains an explicit calculation in ϕ^4 theory that shows that the self-energy is not differentiable at the mass shell.

If one pretends that the self-energy is differentiable at E then the solution to Eq. (4.1) would be

$$\mathcal{E}_{\rm pole}=E+\sum_{s=1}^{\infty}\frac{(-1)^s}{s!}[\Pi_R(E)]^s\left[\frac{d^{s-1}}{dk_0^{s-1}}\frac{1}{[f(k_0)]^s}\right]_{k_0=0},$$

where $f(k_0)$ is the function

$$f(k_0)=-k_0-2E+\sum_{l=1}^{\infty}\frac{(k_0)^{l-1}}{l!}\frac{d^l\Pi_R(E)}{dE^l}. \tag{4.3}$$

However, $d^l\Pi_R(E)/dE^l$ does not exist and thus the perturbative calculation of $\mathcal{E}_{\rm pole}$ fails.

There is another consequence of real dispersion relations that is curious, though perhaps not as dire. Diagrams in which two particles are emitted and none absorbed ($n=2,n'=0$) will have the usual normal threshold branch point at $k_0=2E(\vec{k}/2)$. All contributions in which $n'+2$ particles are emitted and n' are absorbed will also have a branch point at $k_0=2E(\vec{k}/2)$. Similarly, all contributions in which $n'+3$ particles are emitted and n' are absorbed will also have a branch point at $k_0=3E(\vec{k}/3)$. These coincidences will be absent if a complex dispersion relation is employed.

C. Good feature of complex dispersion relations

Any complex dispersion relation (even if the group velocity is real) will generate a perturbative expansion that does not have a branch point in the higher order self-energy at $k_0=\mathcal{E}(k)$ as shown in Sec. III B. Let $\Pi_R^{\rm eff}(k_0)$ be the self-energy computed beyond one-loop order using the free retarded propagator in Eq. (1.5). The radiatively-corrected retarded propagator is

$$D_R^{\rm eff}(k_0)=\frac{1}{(k_0-\mathcal{E})(k_0+\mathcal{E}^*)-\Pi_R^{\rm eff}(k_0)}. \tag{4.4}$$

The pole in this propagator satisfies

$$(\mathcal{E}_{\rm pole}-\mathcal{E})(\mathcal{E}_{\rm pole}+\mathcal{E}^*)=\Pi_R^{\rm eff}(\mathcal{E}_{\rm pole}). \tag{4.5}$$

Since the self-energy $\Pi_R^{\rm eff}(k_0)$ does not have a branch point at $k_0=\mathcal{E}$, it is infinitely differentiable there. The perturbative solution to Eq. (4.5) is

$$\mathcal{E}_{\rm pole}=\mathcal{E}+\sum_{s=1}^{\infty}\frac{(-1)^s}{s!}[\Pi_R^{\rm eff}(\mathcal{E})]^s\left[\frac{d^{s-1}}{dk_0^{s-1}}\frac{1}{[g(k_0)]^s}\right]_{k_0=0},$$

where $g(k_0)$ is the function

$$g(k_0)=-k_0-2\omega+\sum_{l=1}^{\infty}\frac{(k_0)^{l-1}}{l!}\frac{d^l\Pi_R^{\rm eff}(\mathcal{E})}{d\mathcal{E}^l}. \tag{4.6}$$

In this case, $g(k_0)$ does exist and the perturbative expansion for $\mathcal{E}_{\rm pole}$ is valid.

ACKNOWLEDGMENTS

This work was supported in part by National Science Foundation grant PHY-0099380.

APPENDIX A: DETAILED PROOF OF BRANCH POINT AT $k_0=[n-n']E(\vec{k}/[n-n'])$ FOR $n\neq n'$

In Sec. III B it was shown that if the single-particle energy E is real, the necessary conditions for a singularity are satisfied at

$$k_0=(n-n')E(\vec{k}/[n-n']). \tag{A1}$$

This appendix proves sufficiency, viz. that there really is a branch point. The momenta which trap the integration contour are near a saddle point and this makes the analysis different than at zero temperature.

1. Taylor series expansion of ψ near the saddle point

It is convenient to label all the loop momenta as $\vec{p}_j$ so that the denominator function is

$$\psi=\sum_{j=1}^{n}E(\vec{p}_j)-\sum_{j=n+1}^{n+n'}E(\vec{p}_j). \tag{A2}$$

Define $\vec{s}=\vec{k}/(n-n')$ and put

$$\vec{p}_j=\begin{cases}+\vec{s}+\vec{\alpha}_j, & 1\leqslant j\leqslant n,\\ -\vec{s}+\vec{\alpha}_j, & n+1\leqslant j\leqslant n+n'.\end{cases} \tag{A3}$$

The stationary point of ψ that was found in Sec. III B occurs when all $\alpha_j=0$. Momentum conservation requires that

$$0=\sum_{j=1}^{n+n'}\vec{\alpha}_j. \tag{A4}$$

A typical energy can be expanded in a Taylor series to second order in the small quantities $\vec{\alpha}_j$. In doing this it is convenient to decompose the vectors into components parallel and perpendicular to $\vec{k}$ or equivalently to $\vec{s}$. Thus $\vec{\alpha}_j=\hat{k}\alpha_{j\|}+\vec{\alpha}_{j\perp}$. The Taylor series can then be written to second order as

$$E(\vec{s}+\vec{\alpha}_j)=E(\vec{s})+\frac{dE}{ds}\alpha_{j\|}+\frac{1}{2}A_\|\alpha_{j\|}^2+\frac{1}{2}A_\perp\vec{\alpha}_{j\perp}^2+\cdots$$

where

$$A_\|=\frac{d^2E}{ds^2} \tag{A5a}$$

$$A_\perp=\frac{1}{s}\frac{dE}{ds}. \tag{A5b}$$

For a free particle, $E(s)=(s^2+m^2)^{1/2}$, and $A_\|=m^2/E^3$ and $A_\perp=1/E$. For a quasiparticle dispersion relation, one or both of $A_\|$ and $A_\perp$ could be negative. This will not alter the following argument.

When this expansion is inserted into Eq. (A2), the terms linear in $\vec{\alpha}_j$ cancel because of momentum conservation and leave

$$\psi=(n-n')E(s)+\frac{1}{2}A_\|\left(\sum_{j=1}^{n}\alpha_{j\|}^2-\sum_{j=n+1}^{n+n'}\alpha_{j\|}^2\right)+\frac{1}{2}A_\perp\left(\sum_{j=1}^{n}\vec{\alpha}_{j\perp}^2-\sum_{j=n+1}^{n+n'}\vec{\alpha}_{j\perp}^2\right). \tag{A6}$$

The constraint in Eq. (A4) means that there are only $n+n'-1$ linearly independent momentum vectors. One can eliminate the last momentum, $\vec{\alpha}_{n+n'}$, by expressing it as the sum of all the other $\vec{\alpha}$'s. When this is done, Eq. (A6) is no longer diagonal, but it will be real and symmetric and can therefore be diagonalized by a real rotation. The rotation that diagonalizes the terms proportional to $A_\|$ will also diagonalize the terms proportional to $A_\perp$.

2. Diagonalization of ψ

The simplest case is $n=2,n'=1$, i.e. a three particle intermediate state in which two particles are emitted and one is absorbed. From the constraint $\vec{\alpha}_1+\vec{\alpha}_2+\vec{\alpha}_3=0$, express $\vec{\alpha}_3$ in terms of the other two. Then

$$\psi=E(k)+\frac{1}{2}A_\|[\alpha_{1\|}^2+\alpha_{2\|}^2-(\alpha_{1\|}+\alpha_{2\|})^2]+\frac{1}{2}A_\perp[\vec{\alpha}_{1\perp}^2+\vec{\alpha}_{2\perp}^2-(\vec{\alpha}_{1\perp}+\vec{\alpha}_{2\perp})^2]$$

$$=E(k)-A_\|\alpha_{1\|}\alpha_{2\|}-A_\perp\vec{\alpha}_{1\perp}\cdot\vec{\alpha}_{2\perp}.$$

This can be easily diagonalized by defining new momenta $\vec{u}=(\vec{\alpha}_1+\vec{\alpha}_2)/2$ and $\vec{v}=(\vec{\alpha}_1-\vec{\alpha}_2)/2$ so that

$$\psi=E(k)+A_\|(-u_\|^2+v_\|^2)+A_\perp(-u_\perp^2+v_\perp^2).$$

The diagonal form is a function of the six Cartesian components of $\vec{u}$ and $\vec{v}$. The diagonal form is traceless with three positive terms and three negative terms regardless of the signs of $A_\|$ and $A_\perp$.

For general n and n' express $\vec{\alpha}_{n+n'}$ in terms of the other $\vec{\alpha}'s$ using Eq. (A4). Then ψ has the form

$$\psi=(n-n')E(s)-A_\|\sum_{i,j=1}^{n+n'-1}M_{ij}\alpha_{i\|}\alpha_{j\|}-A_\perp\sum_{i,j=1}^{n+n'-1}M_{ij}\vec{\alpha}_{i\perp}\cdot\vec{\alpha}_{j\perp}, \tag{A7}$$

where M_{ij} is a simple numerical matrix. The eigenvalues of this matrix will be called λ_j. The matrix can be diagonalized by a real rotation to a new basis set $\vec{\beta}_j$. In the new basis ψ will be diagonal:

$$\psi=(n-n')E(s)-A_\|\sum_{j=1}^{n+n'-1}\lambda_j\beta_{j\|}^2-A_\perp\sum_{j=1}^{n+n'-1}\lambda_j\vec{\beta}_{j\perp}^2. \tag{A8}$$

None of the eigenvalues λ_j vanish as the following analysis will show.

Case 1. $n'=1$*:* When there is only one absorption $n'=1$ and any number of emissions $n\geqslant 2$, the matrix elements M_{ij} have the following values:

$$M_{ij}=\begin{cases}1, & i\neq j,\\ 0, & i=j.\end{cases} \tag{A9}$$

There are n eigenvalues of this matrix:

$$\lambda=\begin{cases}n-1, & \text{degeneracy}=1,\\ -1, & \text{degeneracy}=n-1.\end{cases} \tag{A10}$$

Thus the diagonal form of ψ is

$$\psi=(n-n')E(s)-A_{\|}\left((n-1)\beta_{\|}^2-\sum_{j=1}^{n-1}\beta_{j\|}^2\right)$$
$$-A_{\perp}\left((n-1)\vec{\beta}_{\perp}^2-\sum_{j=1}^{n-1}\vec{\beta}_{j\perp}^2\right). \tag{A11}$$

Case 2. $n'\geqslant 2$*:* When there are $n'\geqslant 2$ absorptions and a larger number $n>n'$ of emissions, the matrix elements are

$$M_{ij}=\begin{cases}1, & i\neq j,\\ 0, & i=j\leqslant n,\\ 2, & i=j\geqslant n+1.\end{cases} \tag{A12}$$

The matrix has two non-trivial eigenvalues:

$$\lambda_{\pm}=\frac{1}{2}[n+n'-1\pm\sqrt{(n+n'-1)^2-4(n-n')}].$$

Both λ_+ and λ_- are positive. The complete set of eigenvalues are

$$\lambda=\begin{cases}\lambda_+, & \text{degeneracy } =1,\\ \lambda_-, & \text{degeneracy } =1,\\ 1, & \text{degeneracy } =n'-2,\\ -1, & \text{degeneracy } =n-1.\end{cases}$$

Thus n' eigenvalues are positive and $n-1$ are negative.

Thus there are always n' positive eigenvalues, which will be labeled $|\lambda_1|,\ldots,|\lambda_{n'}|$, and $n-1$ negative eigenvalues, which will be labeled $-|\lambda_{n'+1}|,\ldots,-|\lambda_{n'+n-1}|$. Then ψ has the form

$$\psi=(n-n')E(s)-\sum_{j=1}^{n'}|\lambda_j|(A_{\|}\beta_{j\|}^2+A_{\perp}\vec{\beta}_{j\perp}^2)$$
$$+\sum_{j=n'+1}^{n+n'-1}|\lambda_j|(A_{\|}\beta_{j\|}^2+A_{\perp}\vec{\beta}_{j\perp}^2). \tag{A13}$$

The contribution to the retarded self-energy of this small region of momentum space is

$$\Pi_R(k_0)=\int\left(\prod_{j=1}^{n+n'-1}d^3\beta_j\right)\frac{f(\vec{\beta}_j)}{k_0-\psi}. \tag{A14}$$

In the quadratic approximation, ψ does not depend separately on all the vectors $\vec{\beta}_j$ but only on two real variables u and v such that

$$\psi=(n-n')E(s)-u^2+v^2.$$

As noted earlier, $A_{\|}$ and $A_{\perp}$ are expected to be positive. If that is the case, define u and v by

$$u=\left[\sum_{j=1}^{n'}|\lambda_j|(A_{\|}\beta_{j\|}^2+A_{\perp}\vec{\beta}_{j\perp}^2)\right]^{1/2} \tag{A15}$$

$$v=\left[\sum_{j=n'+1}^{n'+n-1}|\lambda_j|(A_{\|}\beta_{j\|}^2+A_{\perp}\vec{\beta}_{j\perp}^2)\right]^{1/2}. \tag{A16}$$

However, if one or both of $A_{\|}$ and $A_{\perp}$ are negative, then define $-u^2$ as the sum of the terms that enter negatively in Eq. (A13) and v^2 as the sum of the terms that enter positively in Eq. (A13). Obviously u and v are real and positive. The relevant integration is only over these two variables and is of the form

$$\int_0^{u_{\max}}du\int_0^{v_{\max}}dv\frac{h(u,v)}{\omega-u^2+v^2}, \tag{A17}$$

where

$$\omega=k_0-(n-n')E(k/[n-n']). \tag{A18}$$

A branch point in the self-energy is now reduced to the question of showing that Eq. (A17) has a branch point at $\omega=0$. The limits $u_{\max}$ and $v_{\max}$ represent the region of validity of the second order Taylor series expansion. The integral as written has a branch cut on the real ω axis for $-v_{\max}^2<\omega<u_{\max}^2$.

3. Existence of the branch point

The question at hand is whether the integral Eq. (A17) has, in addition to the branch cut along the real axis, a branch point at $\omega=0$. The putative existence of such a branch point clearly comes from the region $u\approx v$ and has nothing to do with the upper limits of integration and nothing to do with the numerator function $h(u,v)$. To complete the analysis it is therefore sufficient to examine the function

$$f(\omega)=\int_0^M du\int_0^M dv\,\frac{1}{\omega-u^2+v^2}. \tag{A19}$$

Although this integral cannot be performed explicitly, it is possible to prove the existence of a branch point at $\omega=0$. To analyze this integral it is useful to split the integration over v into two parts:

$$f(\omega)=\int_0^M du\int_0^u dv\,\frac{1}{\omega-u^2+v^2}$$
$$+\int_0^M du\int_u^M dv\,\frac{1}{\omega-u^2+v^2}.$$

In the first integral, replace v by $x=\sqrt{u^2-v^2}$. In the second, replace v by $x=\sqrt{v^2-u^2}$:

$$f(\omega)=\int_0^M du\int_0^u dx\frac{x}{\sqrt{u^2-x^2}}\frac{1}{\omega-x^2}+\int_0^M du\int_0^{\sqrt{M^2-u^2}} dx\frac{x}{\sqrt{u^2+x^2}}\frac{1}{\omega+x^2}.$$

Now interchange the order of integration in both and perform the integrations over u to obtain

$$f(\omega)=\int_0^M dx\left(\frac{x}{\omega-x^2}+\frac{x}{\omega+x^2}\right)\times\ln\left[\frac{M+\sqrt{M^2-x^2}}{x}\right].$$

Note that $f(0)$ is finite, but $[df(\omega)/d\omega]_{\omega=0}$ is divergent as are all the odd derivatives. This already shows that $f(\omega)$ is not analytic at $\omega=0$. [Note that the original function in Eq. (A17) contains a numerator $h(u,v)$ which could vanish at $x=0$. Consequently the divergence of the first derivative of $f(\omega)$ may not hold when the numerator is included. However, higher derivatives will diverge.]

To confirm the branch point at $\omega=0$ the best procedure is to analytically continue ω in a small circle enclosing the origin. The integrand of $f(\omega)$ has simple poles at $x_1=\sqrt{\omega}$, $x_2=-\sqrt{\omega}$, $x_3=i\sqrt{\omega}$, and $x_4=-i\sqrt{\omega}$. When ω has a small, positive imaginary part, these singularities are off the real axis. To expose the branch point at $\omega=0$, set $\omega=re^{i\phi}$. Then as ϕ increases from 0^+ to $2\pi^+$, all four x_j move in small counterclockwise circles and return to different values: x_1 moves to the negative real axis without coming near the integration contour; x_2 moves counterclockwise through the integration contour into the upper half-plane at the position originally occupied by x_1. The change in the value of the integral can be computed by integrating in a small circular contour C_1 around the position originally occupied by x_1:

$$\oint_{C_1} dx\frac{x}{\omega-x^2}\ln\left[\frac{M+\sqrt{M^2-x^2}}{x}\right]=i\pi\ln\left[\frac{M+\sqrt{M^2-\omega}}{\sqrt{\omega}}\right]. \tag{A20}$$

Likewise, as ϕ increases from 0^+ to $2\pi^+$, x_3 moves from the positive imaginary axis clockwise to the negative imaginary axis without touching to the real axis. However, x_4 moves counterclockwise from the negative real axis to the positive real axis and drags the x contour with it. The change in the value of the integral from this distortion can be computed by integrating in a small circular contour C_3 around the position originally occupied by x_3:

$$\oint_{C_3} dx\frac{x}{\omega+x^2}\ln\left[\frac{M+\sqrt{M^2-x^2}}{x}\right]=-i\pi\ln\left[\frac{M+\sqrt{M^2+\omega}}{i\sqrt{\omega}}\right]. \tag{A21}$$

The change in $f(\omega)$ resulting from encircling the origin is the sum of Eqs. (A20) and (A21):

$$f(e^{2\pi i}\omega)-f(\omega)=i\pi\ln\left[\frac{i(M+\sqrt{M^2-\omega})}{M+\sqrt{M^2+\omega}}\right]. \tag{A22}$$

The non-vanishing of the right-hand side confirms that there is a branch point at $\omega=0$ and completes the proof. The branch point has infinitely many sheets because if one rotates the phase of ω by $2\pi N$ the result is

$$f(e^{2\pi Ni}\omega)-f(\omega)=iN\pi\ln\left[\frac{i(M+\sqrt{M^2-\omega})}{M+\sqrt{M^2+\omega}}\right]. \tag{A23}$$

APPENDIX B: DETAILED PROOF OF BRANCH POINT AT $k_0=\pm k$

In Sec. III C it was shown that for a real or complex single-particle energy with the asymptotic behavior in Eq. (3.11) the necessary conditions for a singularity are satisfied at $k_0=\pm k$. This appendix proves that there is a branch point.

As before, it is necessary to examine the integration in the region at which the contour is trapped. The denominator function is

$$\psi=\sum_{j=1}^{n}\mathcal{E}(\vec{p}_j)-\sum_{j=n+1}^{n+n'}\mathcal{E}(\vec{p}_j). \tag{B1}$$

Define the two momenta

$$\vec{s}=\frac{\hat{k}}{n}\left(\frac{k}{2}+P_\parallel\right),\quad \vec{s}\,'=\frac{\hat{k}}{n'}\left(\frac{k}{2}-P_\parallel\right), \tag{B2}$$

and set

$$\vec{p}_j=\begin{cases}\vec{s}+\vec{\alpha}_j\sqrt{P_\parallel}, & 1\leqslant j\leqslant n,\\ \vec{s}\,'+\vec{\alpha}_j\sqrt{P_\parallel}, & n+1\leqslant j\leqslant n+n'.\end{cases} \tag{B3}$$

Momentum conservation requires that

$$0=\sum_{j=1}^{n+n'}\vec{\alpha}_j. \tag{B4}$$

When all the $\vec{\alpha}_j=0$

$$\psi|_{\vec{\alpha}_j=0}=n\mathcal{E}(\vec{s})-n'\mathcal{E}(\vec{s}\,').$$

To demonstrate that there is a branch point it is necessary to expand ψ in a Taylor series for $|\vec{\alpha}_j|\sqrt{P_\parallel}$ small compared to $|\vec{s}|$ and $|\vec{s}'|$:

$$\psi = nE(s) + \frac{P_\parallel}{2}\sum_{j=1}^{n}(A_\parallel \alpha_{j\parallel}^2 + A_\perp \vec{\alpha}_{j\perp}^2)$$

$$-n'E(s') - \frac{P_\parallel}{2}\sum_{j=n+1}^{n+n'}(A'_\parallel \alpha_{j\parallel}^2 + A'_\perp \vec{\alpha}_{j\perp}^2).$$

The terms linear in $\vec{\alpha}_j$ canceled by momentum conservation as in Appendix A. The coefficients $A_\parallel$ and $A_\perp$ are as defined in Eq. (A5).

The branch point we are seeking occurs when $P_\parallel \to \infty$. In this limit both s and s' approach infinity so that

$$\mathcal{E}(s) \to s + \frac{m^2}{2s} + \cdots$$

$$\frac{d\mathcal{E}}{ds} \to 1 - \frac{m^2}{2s^2} + \cdots$$

$$A_\parallel \to \frac{m^2}{3s^2} + \cdots$$

$$A_\perp \to \frac{1}{s} - \frac{m^2}{2s^3} + \cdots.$$

At large $P_\parallel$ the denominator function ψ behaves as

$$\psi \to k + \frac{(mn)^2}{2P_\parallel + k} - \frac{(mn')^2}{2P_\parallel - k}$$

$$+ \frac{nP_\parallel}{2P_\parallel + k}\sum_{j=1}^{n}\vec{\alpha}_{j\perp}^2 - \frac{n'p_\parallel}{2P_\parallel - k}\sum_{j=n+1}^{n+n'}\vec{\alpha}_{j\perp}^2.$$

Now take $P_\parallel \to \infty$ so that

$$P_\parallel \to \infty: \quad \psi = k + \frac{n}{2}\sum_{j=1}^{n}\vec{\alpha}_{j\perp}^2 - \frac{n'}{2}\sum_{j=n+1}^{n+n'}\vec{\alpha}_{j\perp}^2. \quad \text{(B5)}$$

This is of the same form as Eq. (A6) except that $\vec{\alpha}_{j\parallel}$ does not enter. Therefore, the proof from Appendix A applies. The contribution to the retarded self-energy is

$$\Pi_R(k_0) = \int \left(\prod_{j=1}^{n+n'} d^3\alpha_j\right)\frac{f(\vec{\alpha}_j)}{k_0 - \psi}. \quad \text{(B6)}$$

The proof in Appendix A shows that if $k_0 = k + re^{i\phi}$ then the value of $\Pi_R(k_0)$ does not return to the same value when ϕ increases from 0 to 2π.

APPENDIX C: ϕ^3 EXAMPLE TO ONE LOOP

The simplest example of an essential singularity at $k_0 = \pm k$ occurs in a theory with $\mathcal{H}_I = g\phi^3/3!$. With the free-particle dispersion relation $E(p) = (p^2 + m^2)^{1/2}$, the one-loop self-energy has a branch cut for $-k \leqslant k_0 \leqslant k$ that results from an intermediate state with $n = n' = 1$. The following calculation will expose the essential singularity at $k_0 = \pm k$ in the self-energy:

$$\Pi_R(k_0) = g^2\int \frac{d^3p}{(2\pi)^3}\frac{n(p) - n(\vec{p}+\vec{k})}{(k_0 + i\epsilon - \psi)2E(p)2E(\vec{p}+\vec{k})}.$$

The denominator function is

$$\psi = E(\vec{p}+\vec{k}) - E(p).$$

Rather that calculate Π_R itself, it is easier to calculate the imaginary part:

$$\mathrm{Im}\,\Pi_R(K) = -\frac{g^2\pi}{4}\int \frac{d^3p}{(2\pi)^3}\delta[k_0 - \psi]\frac{n(p) - n(\vec{p}+\vec{k})}{E(p)E(\vec{p}+\vec{k})}.$$

With the decomposition $\vec{p} = \hat{k}p_\parallel + \vec{p}_\perp$, the integral over $\vec{p}_\perp$ can be performed using the Dirac delta function:

$$\int d^2p_\perp\,\delta[k_0 - \psi] = \frac{2\pi p_\perp}{d\psi/dp_\perp}\Big|_{\psi = k_0} = 2\pi\frac{E(\vec{p})E(\vec{p}+\vec{k})}{k_0}.$$

Consequently

$$\mathrm{Im}\,\Pi_R(K) = -\frac{g^2}{16\pi k_0}\int_{p_\parallel^{\min}}^{\infty} dp_\parallel[n(p) - n(p+k)]. \quad \text{(C1)}$$

The condition $k_0 = \psi$ can only be satisfied for $K^2 < 0$ and it makes $E(\vec{p})$ a linear function of $p_\parallel$:

$$E(\vec{p}) = \frac{k}{k_0}p_\parallel - \frac{K^2}{2k_0},$$

where

$$p_\parallel = -\frac{k}{2} + \frac{k_0}{2}\sqrt{1 - \frac{4(m^2 + p_\perp^2)}{K^2}}.$$

Since $0 \leqslant p_\perp \leqslant \infty$, the minimum value of the parallel momentum is

$$p_\parallel^{\min} = -\frac{k}{2} + \frac{k_0}{2}\sqrt{1 - \frac{4m^2}{K^2}}.$$

The remaining integration in Eq. (C1) is elementary:

$$\mathrm{Im}\,\Pi_R(K) = -\frac{g^2 T}{16\pi k}\ln\left[\frac{1 - e^{-\beta E(k + p_\parallel^{\min})}}{1 - e^{-\beta E(p_\parallel^{\min})}}\right], \quad \text{(C2)}$$

where the energies that enter are

$$E(p_\parallel^{\min}) = -\frac{k_0}{2} + \frac{k}{2}\sqrt{1-\frac{4m^2}{K^2}}$$

$$E(k+p_\parallel^{\min}) = \frac{k_0}{2} + \frac{k}{2}\sqrt{1-\frac{4m^2}{K^2}}.$$

Both energies are positive since $K^2<0$. Thus the imaginary part is

$$\mathrm{Im}\,\Pi_R(K) = -\frac{g^2T}{16\pi k}\ln\left[\frac{1-e^{-\beta(k_0+k\sqrt{1-4m^2/K^2})/2}}{1-e^{-\beta(-k_0+k\sqrt{1-4m^2/K^2})/2}}\right].$$

The imaginary part is an odd function of k_0. As expected, there is an essential singularity at $k_0=\pm k$. The leading behavior as $k_0\to k$ is

$$\mathrm{Im}\,\Pi_R(K) \to -\frac{g^2T}{16\pi k}(e^{\beta k/2}-e^{-\beta k/2}) \times \exp\left(-\frac{\beta k}{2}\sqrt{1-\frac{2(nm)^2}{k(k_0-k)}}\right). \quad \text{(C3)}$$

This agrees perfectly with Eq. (3.23) for $n=n'=1$.

APPENDIX D: ϕ^4 EXAMPLE AT TWO LOOPS

This appendix will explicitly show the branch points on the light cone and on the mass shell in ϕ^4 theory at two-loop order. The analysis of this section will be based on the work of Wang and Heinz [19], who calculated the imaginary part of the self-energy to two-loop order as a function of energy. Previous works had computed the imaginary part just on the mass-shell [21,22].

The notation in this appendix will be that of Wang and Heinz [19]. The interaction Hamiltonian is $g^2\phi^4/4!$. The zero-temperature particles are taken as massless, but thermal resummation leads to propagators with poles at $p_0=\pm(p^2+m_P^2)^{1/2}$, where m_P is a resummed plasmon mass

$$m_P^2 = \frac{g^2T^2}{24}\left(1-\frac{g}{2\pi}\sqrt{\frac{3}{2}}\right). \quad \text{(D1)}$$

The imaginary part of the two-loop self-energy is grouped as

$$\mathrm{Im}\,\Sigma(\omega,\vec{p}) = \mathrm{Im}\,g_1(\omega,\vec{p}) + \mathrm{Im}\,g_2(\omega,\vec{p}).$$

Here g_1 contains the usual three-particle cuts (i.e. $n=3$, $n'=0$ and $n=0$, $n'=3$) and will not be discussed here; g_2 contains the cut for two emissions and one absorption ($n=2,n'=1$) and the cut for one emission and two absorption ($n=1,n'=2$). As demonstrated in Sec. III, the function $g_2(\omega,\vec{p})$ should have branch points at $\omega=\pm(p^2+m_P^2)^{1/2}$ and essential singularities at $\omega=\pm p$. The results of Wang and Heinz for $\mathrm{Im}\,g_2(\omega,\vec{p})$ are extremely complicated double integrals when $\vec{p}\neq 0$. Consequently, this appendix will only examine $\vec{p}=0$ and will demonstrate a branch point at $\omega=m_P$ and an essential singularity at $\omega=0$.

At $\vec{p}=0$, Wang and Heinz express the imaginary part of the self-energy as

$$\omega<m_P: \quad \mathrm{Im}\,g_2(\omega,0) = \int_\varepsilon^\infty dv\,F(w,v) \quad \text{(D2)}$$

$$\omega>m_P: \quad \mathrm{Im}\,g_2(\omega,0) = \int_a^\infty dv\,F(w,v). \quad \text{(D3)}$$

In the integrals, v and w are Latin letters: v is a dimensionless variable (an energy divided by T) and w is the dimensionless ratio

$$w=\frac{\omega}{T}.$$

The lower limit of the first integral is

$$\varepsilon = \left[a^2 + \frac{(a^2-w^2)(9a^2-w^2)}{4w^2}\right]^{1/2}, \quad \text{(D4)}$$

and the lower limit of the second integral is

$$a=\frac{m_P}{T}.$$

Note that $\varepsilon\to a$ when $w\to a$.

1. Branch point at $\omega=m_P$

At the mass shell $\omega=m_P$ (equivalently $w=a$) the lower limits are equal ($\varepsilon=a$) and thus $\mathrm{Im}\,g_2(\omega,0)$ is continuous at $\omega=m_P$.

The first derivative of Eq. (D2) is

$$\omega<m_P: \quad T\frac{d\,\mathrm{Im}\,g_2(\omega,0)}{d\omega} = -\frac{d\varepsilon}{dw}F(w,\varepsilon) + \int_\varepsilon^\infty dv\,\frac{\partial F(w,v)}{\partial w}. \quad \text{(D5)}$$

An essential property of the integrand is [19]

$$F(w,v)\big|_{v=a} = 0. \quad \text{(D6)}$$

As $\omega\to m_P$, the lower limit $\varepsilon\to a$. Since $F(a,a)=0$ by Eq. (D6), the first derivative of Eq. (D2) at $\omega=m_P$ is the same as the first derivative of Eq. (D3) at $\omega=m_P$.

The second derivative of Eq. (D2) is

$$\omega<m_P: \quad T^2\frac{d^2\mathrm{Im}\,g_2(\omega,0)}{d\omega^2}=-\frac{d}{dw}\left[\frac{d\varepsilon}{dw}F(w,\varepsilon)\right] - \frac{d\varepsilon}{dw}\frac{\partial F(w,\varepsilon)}{\partial w} + \int_\varepsilon^\infty dv\,\frac{\partial^2 F(w,v)}{\partial w^2}. \tag{D7}$$

The second term on the right-hand side vanishes at $w=a$ because of Eq. (D6). The first term on the right-hand side simplifies to

$$-\frac{d^2\varepsilon}{dw^2}F(w,\varepsilon)-\frac{d\varepsilon}{dw}\frac{\partial F(w,\varepsilon)}{\partial w}-\left[\frac{d\varepsilon}{dw}\right]^2\frac{\partial F(w,\varepsilon)}{d\varepsilon}.$$

The first two terms of this vanish at $w=a$ because of Eq. (D6), but the third does not. Thus as ω approaches m_P from below, the second derivative is

$$T^2\frac{d^2\mathrm{Im}\,g_2(\omega,0)}{d\omega^2}\bigg|_{\omega=m_P^-}=-\left[\frac{d\varepsilon}{dw}\right]^2_{\omega=a}\frac{\partial F(a,v)}{\partial v}\bigg|_{v=a} + \int_a^\infty dv\,\frac{\partial^2 F(w,v)}{\partial w^2}. \tag{D8}$$

At $\omega=a$, Eq. (D4) gives $d\varepsilon/dw=-2$. From Wang and Heinz [19] the function $F(w,v)$ simplifies at $w=a$ to

$$F(a,v)=\frac{g^4T^2}{128\pi^3}\frac{e^v}{e^v-1}\frac{e^a-1}{e^{a+v}-1}2\ln\left[\frac{\sinh(v/2)}{\sinh(a/2)}\right]. \tag{D9}$$

As expected, this vanishes at the lower limit $v=a$. However, $\partial F(a,v)/\partial v$ does not vanish at $v=a$. Thus the second derivative is discontinuous at $w=m_P$ with a discontinuity given by

$$\left[\frac{d^2\mathrm{Im}\,g_2(\omega,0)}{d\omega^2}\right]^{\omega=m_P^+}_{\omega=m_P^-}=\frac{g^4}{32\pi^3}\frac{e^a}{(e^a-1)^2}. \tag{D10}$$

This confirms the existence of a branch point at the mass shell $\omega=m_P$.

There is a further check of Eq. (D10). In the specific calculation of Wang and Heinz, the mass was entirely thermal so that $a=m_P/T$ is independent of temperature. However, the calculation of the two-loop discontinuity would also apply in a theory with a non-thermal mass m. Then the right-hand side of Eq. (D10) would be temperature dependent with $a=m/T$. In the zero-temperature limit, $a\to\infty$ and discontinuity in the second derivative vanishes as expected.

2. Essential singularity at $\omega=0$

For $\vec{p}\neq 0$, the function $g_2(\omega,\vec{p})$ will have essential singularities at $\omega=\pm p$. In the case considered here, viz. $\vec{p}=0$, these collapse to an essential singularity at $\omega=0$. In the vicinity of $\omega\approx 0$, the imaginary part is given by Eq. (D2). The lower limit of the integral grows as $\omega\to 0$:

$$\omega\to 0: \quad \varepsilon\to\frac{3a^2}{2w}+\cdots.$$

It is convenient to change variables from v to $\bar{v}$,

$$v=\frac{3a^2}{2w}[1+\bar{v}], \tag{D11}$$

where $0\leqslant\bar{v}\leqslant\infty$. Then Eq. (D2) becomes

$$\omega<m_P: \quad \mathrm{Im}\,g_2(\omega,0)=\frac{3a^2}{2w}\int_0^\infty d\bar{v}F(w,v). \tag{D12}$$

From [19] the integral becomes in the limit $\omega\to 0$:

$$\mathrm{Im}\,g_2(\omega,0)\to\frac{9g^4T^2}{1024\pi^3}\frac{a^4}{w}e^{-3a^2/2w}I(\omega) \tag{D13}$$

$$I(\omega)=\int_0^\infty d\bar{v}e^{-3a^2\bar{v}/2\omega}(1+\bar{v})\left(1+\sqrt{\frac{3\bar{v}}{4+3\bar{v}}}\right).$$

As $w\to 0$, the integrand of $I(\omega)$ is exponentially small for any $\bar{v}$ that is not infinitesimal. The dominant contribution comes from the region $0\leqslant\bar{v}\ll 2w/3a^2$ and gives $I(\omega)\to 2\omega/3a^2$. Thus

$$\omega\to 0: \quad \mathrm{Im}\,g_2(\omega,0)\to\frac{3g^4m_P^2}{512\pi^3}e^{-3a^2/2w}. \tag{D14}$$

The exponent here, $-3a^2/2w$, agrees precisely with that anticipated in Eq. (3.24) for $n=2,n'=1$.

[1] R.D. Pisarski, Nucl. Phys. **A498**, 423c (1989); Phys. Rev. Lett. **63**, 1129 (1989).

[2] E. Braaten and R.D. Pisarski, Nucl. Phys. **B337**, 569 (1990); **B339**, 310 (1990).

[3] T.S. Evans, Nucl. Phys. **B374**, 340 (1992).

[4] J. Bros and D. Buchholz, Ann. I.H.P. Phys. Theor. **64**, 495 (1996).

[5] P.S. Gribosky and B.R. Holstein, Z. Phys. C **47**, 205 (1990); P.F. Bedaque and A. Das, Phys. Rev. D **45**, 2906 (1992); **47**, 601 (1993); H.A. Weldon, *ibid.* **47**, 594 (1993); P. Arnold, S. Vokos, P. Bedaque, and A. Das, *ibid.* **47**, 4698 (1993); A. Das and M. Hott, *ibid.* **50**, 6655 (1994); T.S. Evans, Nucl. Phys.

B496, 486 (1997); F.T. Brandt, A. Das, J. Frenkel, and J.C. Taylor, Phys. Rev. D (to be published), hep-th/0112016.
[6] M. Le Bellac, *Thermal Field Theory* (Cambridge University Press, Cambridge, England, 1996).
[7] A. Das, *Finite Temperature Field Theory* (World Scientific, Singapore, 1997).
[8] P. Aurenche and T. Becherrawy, Nucl. Phys. **B379**, 259 (1992).
[9] M.A. van Eijck, R. Kobes, and Ch.G. van Weert, Phys. Rev. D **50**, 4097 (1994).
[10] J.I. Kapusta, *Finite-Temperature Field Theory* (Cambridge University Press, Cambridge, England, 1989).
[11] H.A. Weldon, Phys. Rev. D **28**, 2007 (1983).
[12] J.P. Blaizot, J.Y. Ollitrault, and E. Iancu, in *Quark-Gluon Plasma*, edited by R.C. Hwa (World Scientific, Singapore, 1995), pp. 171 and 191.
[13] A.V. Smilga, Phys. Rep. **291**, 1 (1977).
[14] H.A. Weldon, Phys. Rev. D **58**, 105002 (1998).
[15] R.J. Eden, P.V. Landshoff, D.I. Olive, and J.C. Polkinghorne, *The Analytic S Matrix* (Cambridge University Press, Cambridge, England, 1966).
[16] C. Itzykson and J.B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980), pp. 301–312.
[17] R. Kobes and G. Semenoff, Nucl. Phys. **B260**, 714 (1985); **B272**, 329 (1986); P.F. Bedaque. A. Das, and S. Naik, Mod. Phys. Lett. A **12**, 2481 (1997); P.V. Landshoff, Phys. Lett. B **386**, 291 (1996); S.M.H. Wong, Phys. Rev. D **64**, 025007 (2001).
[18] R. Baier, H. Nakkagawa, and A. Niégawa, Can. J. Phys. **71**, 205 (1993); S. Peigné, E. Pilon, and D. Schiff, Z. Phys. C **60**, 455 (1993); A.V. Smilga, Phys. At. Nucl. **57**, 519 (1994).
[19] E. Wang and U. Heinz, Phys. Rev. D **53**, 899 (1996).
[20] P.A. Henning, E. Poliatchenko, T. Schilling, and J. Bros, Phys. Rev. D **54**, 5239 (1996).
[21] R.R. Parwani, Phys. Rev. D **45**, 4695 (1992).
[22] S. Jeon, Phys. Rev. D **52**, 3591 (1995).